

PHIL 4310: Advanced Logic (Spring 2026)

Example from class today:

Distribution laws:

$$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$$

$$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$$

Lets prove just one of these four examples: $A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$

To do this, we will first construct a Fitch style natural deduction. Then I will do it in a Lemmon style proof, then finally, a Sider style sequent proof.

EXAMPLE 1 $A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$

Step 1. Start by writing the premise at the top and the conclusion at the bottom. Since our premise is an \vee , I will set up an \vee Elim. This requires creating two subproofs - one that starts with the left disjunct and ends with your goal and the other that starts with the right disjunct and ends with your goal. When you make an assumption it is usually a good idea to make a notation to yourself how you plan to discharge it.

1)	$A \vee (B \wedge C)$	As
2)	A	As
...		
j)	$(A \vee B) \wedge (A \vee C)$	goal
k)	$B \wedge C$	As
...		
m)	$(A \vee B) \wedge (A \vee C)$	goal
n)	$(A \vee B) \wedge (A \vee C)$	\vee Elim

Step 2. Within the first subproof, we have a conjunction as a goal. So clearly we need to use \wedge I to get it. Each part that we need is then a disjunction. To get those, we will use \vee I. In the second subproof, first we have a conjunction ($B \wedge C$). So use \wedge E to break it up. Now our goal is a conjunction, so we need to use \wedge I to get it. Again, \vee I works. Now that both subproofs are complete, we can prove our goal outside of any subproof by using the \vee Elim rule.

1)	$A \vee (B \wedge C)$	As
2)	A	As (for \vee E)
3)	$A \vee B$	\vee I 2
4)	$A \vee C$	\vee I 2
5)	$(A \vee B) \wedge (A \vee C)$	\wedge I 3,4
6)	$B \wedge C$	As (for \vee E)
7)	B	\wedge E 6
8)	C	\wedge E 6
9)	$A \vee B$	\vee I 7
10)	$A \vee C$	\vee I 8
11)	$(A \vee B) \wedge (A \vee C)$	\wedge I 9,10
12)	$(A \vee B) \wedge (A \vee C)$	\vee Elim 1,2-5,6-11

I will now do this exact proof using a Lemmon style proof system with assumption tracking.

Step 1. Start by writing the premise at the top and the conclusion at the bottom. Since our premise is an \vee , I will set up an \vee Elim. This requires creating two subproofs - one that starts with the left disjunct and ends with your goal and the other that starts with the right disjunct and ends with your goal. When you make an assumption it is usually a good idea to make a notation to yourself how you plan to discharge it.

1	1)	$A \vee (B \wedge C)$	As
2	2)	A	As (for \vee E)
	...		
	j)	$(A \vee B) \wedge (A \vee C)$	goal
	k)	$B \wedge C$	As (for \vee E)
	...		
	m)	$(A \vee B) \wedge (A \vee C)$	goal
1	n)	$(A \vee B) \wedge (A \vee C)$	\vee Elim

Step 2. Within the first subproof, we have a conjunction as a goal. So clearly we need to use \wedge I to get it. Each part that we need is then a disjunction. To get those, we will use \vee I. In the second subproof, first we have a conjunction ($B \wedge C$). So use \wedge E to break it up. Now our goal is a conjunction, so we need to use \wedge I to get it. Again, \vee I works. Now that both subproofs are complete, we can prove our goal outside of any subproof by using the \vee Elim rule.

1	1)	$A \vee (B \wedge C)$	As
2	2)	A	As (for \vee E)
2	3)	$A \vee B$	\vee I 2
2	4)	$A \vee C$	\vee I 2
2	5)	$(A \vee B) \wedge (A \vee C)$	\wedge I 3,4
6	6)	$B \wedge C$	As (for \vee E)
6	7)	B	\wedge E 6
6	8)	C	\wedge E 6
6	9)	$A \vee B$	\vee I 7
6	10)	$A \vee C$	\vee I 8
6	11)	$(A \vee B) \wedge (A \vee C)$	\wedge I 9,10
1	12)	$(A \vee B) \wedge (A \vee C)$	\vee Elim 1,5,11,(2),(6)

Notice that the assumptions depend on themselves and that rules like \vee I, \wedge I that combine previous lines depend on every assumption that those lines depend on. Rules like \vee Elim allow you to discharge assumptions, In the Fitch style system, we rely on geometric scope lines to indicate when we are inside the scope of an assumption and when we are no longer inside. In the Lemmon style system, for each line we keep track of which assumptions we needed to derive that line in particular.

I will now do a Sider style sequent proof of the problem. The easiest way is to take the Lemmon style proof and simply replace the assumptions on the far left with the actual lines. The \vee Elim rule is also slightly different in terms of line citations.

- 1) $A \vee (B \wedge C) \Rightarrow A \vee (B \wedge C)$ As
- 2) $A \Rightarrow A$ As (for \vee E)
- 3) $A \Rightarrow (A \vee B)$ \vee I 2
- 4) $A \Rightarrow (A \vee C)$ \vee I 2
- 5) $A \Rightarrow (A \vee B) \wedge (A \vee C)$ \wedge I 3,4
- 6) $(B \wedge C) \Rightarrow (B \wedge C)$ As (for \vee E)
- 7) $(B \wedge C) \Rightarrow B$ \wedge E 6
- 8) $(B \wedge C) \Rightarrow C$ \wedge E 6
- 9) $(B \wedge C) \Rightarrow (A \vee B)$ \vee I 7
- 10) $(B \wedge C) \Rightarrow (A \vee C)$ \vee I 8
- 11) $(B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$ \wedge I 9,10
- 12) $A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$ \vee Elim 1,5,11